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Constants of motion, ladder operators and supersymmetry of the two-dimensional isotropic harmonic oscillator

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Abstract
For the quantum two-dimensional isotropic harmonic oscillator we show that the Infeld–Hull radial operators, as well as those of the supersymmetric approach for the radial equation, are contained in the constants of motion of the problem.

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1. Introduction

For some quantum problems the relation between constants of motion, ladder operators and supercharges has been studied by several authors. For example, in their seminal work Infeld and Hull showed the connection between factorization operators and the orbital angular momentum [1]. The Infeld–Hull (IH) factorization can be derived from the ladder operators related to the angular momentum operators in the symmetric top, the electron–magnetic pole interaction and Weyl's spherical harmonics with spin [2]. Besides, the Laplace–Runge–Lenz vector (LRLV) has been related to the Infeld–Hull factorization method (IHFM) [2–4] and to the pair of isospectral Hamiltonians for the non-relativistic Kepler–Coulomb quantum problem [4–6]. Also, in [7] it is shown that the supersymmetry of the relativistic Kepler quantum problem is generated by the Johnson–Lippmann operator.

We can ask ourselves for other quantum problems where there is any connection between the constants of motion and their radial ladder operators. The purpose of this paper is to answer this question by treating the two-dimensional isotropic harmonic oscillator (2-DIHO) quantum problem. To do this we follow the approach developed in [4], the radial Schrödinger equation factorized *a lá* IH, and the relationship between degeneracy, constants of motion and the $SU(2)$ symmetry group for the 2-DIHO.

In section 2, we find, from the classical observables of the problem, the quantum operators $S_{\pm}$, which connect all the states with a given energy. We show how these quantum operators are related with the left and right circular annihilation operators. We apply them to any wavefunction of the function space spanned by the solutions of the Schrödinger equation. This procedure allows us to obtain the radial ladder operators for the reduced wavefunction $f_{nm}(\rho)$, acting on m, without reference to any factorization method. Next, by using the radial Schrödinger equation we show that these operators are equal to those we obtain by using the IHFM. So, we conclude that the IH radial operators of the problem are contained in the quantum operators $S_{\pm}$.

In section 3, by means of simple arguments, we show how the ladder operators obtained in section 2 must generate the pair of isospectral Hamiltonians of the 2-DIHO. Finally, in section 4, we give some concluding remarks.

2. The constants of motion $S_{\pm}$, annihilation operators a_d, a_g and ladder operators for $f_{nm}(\rho)$, acting on m

It is well known that the constants of motion of the classical 2-DIHO (in addition to the angular momentum L_z) are given by the symmetric tensor of rank 2 [8]:

$$A_{ij} = \frac{1}{2\mu}\left(P_i P_j + \mu^2\omega^2 x_i x_j\right) \tag{1}$$

with $\mu(\omega)$ the mass (frequency), $\boldsymbol{P}$ the linear momentum of the oscillator and $i, j = x, y$. The quantum version of this tensor follows immediately, since it turns out to be self-adjoint. These operators and the angular momentum L_z are used to define the new constants of motion:

$$S_x \equiv (A_{xy} + A_{yx})/2\omega = \left(P_x P_y + \mu^2\omega^2 xy\right)/2\mu\omega \tag{2}$$

$$S_y \equiv (A_{yy} - A_{xx})/2\omega = \left(P_y^2 - P_x^2 + \mu^2\omega^2(y^2 - x^2)\right)/4\mu\omega \tag{3}$$

$$S_z \equiv L_z/2 = (xP_y - yP_x)/2. \tag{4}$$

Except for some small changes, these are those given in [9]. S_x is known as the correlation and is a peculiar feature of the 2-DIHO [10]. S_y is the energy difference of the one-dimensional harmonic oscillators in the x- and y-coordinates. They obey the $SU(2)$ or $SO(3)$ group Lie algebra:

$$[S_i, S_j] = \mathrm{i}\hbar\epsilon_{ijk}S_k \qquad \text{for} \quad i, j, k = x, y, z. \tag{5}$$

These commutation relations leads to

$$[L_z, S_{\pm}] = \pm 2\hbar S_{\pm} \tag{6}$$

where $S_{\pm} = S_x \pm \mathrm{i}S_y$.

The Hamiltonian of the 2-DIHO is

$$H = \frac{1}{2\mu}\left(P_x^2 + P_y^2 + \mu^2\omega^2(x^2 + y^2)\right) \tag{7}$$

or, in polar coordinates,

$$H = -\frac{\hbar^2}{2\mu}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right) + V(\rho) \tag{8}$$

where $V(\rho) = \mu\omega^2\rho^2/2$. Because of $[A_{ij}, H] = [L_z, H] = 0$, it is straightforward to show that $[H, S_i] = 0$.

We know that the operators $\{H, L_z\}$ are a complete set of commuting observables in the state space ξ_{xy} associated with the variables x and y [11]. Then by applying equation (6) to any eigensolution ψ_{nm} of the eigenvalue equations

$$H\psi(\rho,\phi) = E\psi(\rho,\phi) \tag{9}$$

and

$$L_z\psi(\rho,\phi) = \hbar m\psi(\rho,\phi) \tag{10}$$

we obtain

$$L_z S_\pm\psi_{nm} = S_\pm(L_z \pm 2\hbar)\psi_{nm} = \hbar(m \pm 2)S_\pm\psi_{nm}. \tag{11}$$

Since this expresses the fact that $S_\pm\psi_{nm}$ is also an eigenfunction of L_z with eigenvalues $m \pm 2$, then

$$S_\pm\psi_{nm} \propto \psi_{n\,m\pm2} \tag{12}$$

i.e. S_+ (S_-) increases (decreases) two units in m, leaving the principal quantum number n fixed, when it acts over the complete state $\psi_{nm}(\rho,\phi)$. Therefore, the degeneracy of the 2-DIHO can be described by using the operators $S_\pm$.

Equation (12) suggests a connection between the operators $S_\pm$ and the left and right circular annihilation operators a_d and a_g defined as [11]

$$a_d = \frac{1}{\sqrt{2}}(a_x - \mathrm{i}a_y) \tag{13}$$

$$a_g = \frac{1}{\sqrt{2}}(a_x + \mathrm{i}a_y) \tag{14}$$

where

$$a_x = \frac{1}{\sqrt{2}}(\beta x + \mathrm{i}P_x/\beta\hbar) \tag{15}$$

$$a_y = \frac{1}{\sqrt{2}}(\beta y + \mathrm{i}P_y/\beta\hbar) \tag{16}$$

with $\beta = \sqrt{\mu\omega/\hbar}$. Since the action of the operators a_d and a_g on any 2-DIHO wavefunction ψ_{nm} leads us to

$$a_d\psi_{nm} = \sqrt{\tfrac{1}{2}(n+m)}\psi_{n-1\,m-1} \qquad a_d^\dagger\psi_{nm} = \sqrt{\tfrac{1}{2}(n+m)+1}\psi_{n+1\,m+1} \tag{17}$$

$$a_g\psi_{nm} = \sqrt{\tfrac{1}{2}(n-m)}\psi_{n-1\,m+1} \qquad a_g^\dagger\psi_{nm} = \sqrt{\tfrac{1}{2}(n-m)}\psi_{n+1\,m-1} \tag{18}$$

where the dagger implies Hermitian conjugate, we deduce that

$$\begin{aligned} a_d^\dagger a_g\psi_{nm} &= \sqrt{n_g(n_d+1)}\psi_{n\,m+2} \\ a_g^\dagger a_d\psi_{nm} &= \sqrt{n_d(n_g+1)}\psi_{n\,m-2} \end{aligned} \tag{19}$$

with $n_d = (n+m)/2$ and $n_g = (n-m)/2$.

Thus, from equations (12) and (19) we can say that S_+ (S_-) is a multiple of $a_d^\dagger a_g$ ($a_g^\dagger a_d$). In fact, we easily verify that

$$\begin{aligned} S_+ &= -\mathrm{i}a_d^\dagger a_g \\ S_- &= +\mathrm{i}a_g^\dagger a_d. \end{aligned} \tag{20}$$

On the other hand, for all two-dimensional central potentials $V(\rho)$, ψ_{nm} can be written as

$$\psi_{nm} = \mathrm{e}^{\mathrm{i}m\phi} R_{nm}(\rho). \tag{21}$$

Now, we want to show how the IH radial ladder operators for the reduced wavefunction

$$f_{nm}(\rho) \equiv \rho R_{nm}(\rho) \tag{22}$$

acting on m, are contained in $S_\pm$. By expressing the operators $S_\pm$ in polar coordinates we find

$$S_\pm = \frac{-\hbar^2}{4\mu\omega}\,\mathrm{e}^{\pm 2\phi}\left(\mp\mathrm{i}\left(\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} - \frac{\mu^2\omega^2}{\hbar^2}\rho^2\right) - \frac{2}{\rho^2}\frac{\partial}{\partial\phi} + \frac{2}{\rho}\frac{\partial^2}{\partial\rho\partial\phi}\right). \tag{23}$$

Applying them to any bound state of the form (21), we obtain

$$S_\pm\psi_{nm} = \pm\frac{\mathrm{i}\hbar^2}{4\mu\omega}\,\mathrm{e}^{\mathrm{i}(m\pm 2)\phi}\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{1}{\rho}(\pm 2m+1)\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{m^2 \pm 2m}{\rho^2} - \frac{\mu^2\omega^2}{\hbar^2}\rho^2\right) R_{nm}(\rho). \tag{24}$$

The second-order derivative in this equation can be transformed into one of first order by using the radial Schrödinger equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{m^2}{\rho^2} - \beta^4\rho^2\right) R_{nm}(\rho) = -\beta^2\gamma R_{nm}(\rho) \tag{25}$$

with $\gamma = 2E/\hbar\omega$. Then we obtain

$$S_\pm\psi_{nm} = -\frac{\mathrm{i}\hbar}{2\beta^2}(m\pm 1)\,\mathrm{e}^{\mathrm{i}(m\pm 2)\phi}\left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} \mp \frac{m}{\rho^2} \pm \frac{\gamma}{2(m\pm 1)}\beta^2\right) R_{nm}(\rho) \tag{26}$$

which is of first order. An important fact that will be used in section 3 is that these equations are not defined for $m = \mp 1$. Then, because of equations (12) and equation (26), the explicit first-order radial ladder operators, the step-up and step-down in m must be such that

$$\left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} \mp \frac{m}{\rho^2} \pm \frac{\gamma}{2(m\pm 1)}\beta^2\right) R_{nm}(\rho) = R_{n\,m\pm 2}(\rho). \tag{27}$$

Equivalently, if we use the reduced radial wavefunction (22), equation (26) takes the form

$$S_\pm\psi_{nm} = -\frac{\mathrm{i}\hbar}{2\beta^2}(m\pm 1)\,\mathrm{e}^{\mathrm{i}(m\pm 2)\phi}\frac{1}{\rho}\left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{1\pm m}{\rho^2} \pm \frac{\gamma}{2(m\pm 1)}\beta^2\right) f_{nm}(\rho). \tag{28}$$

Performing the change of variable $\tilde{x} = \beta^2\rho^2$, this equation can be rewritten as

$$S_\pm\psi_{nm} = \pm\mathrm{i}\hbar\beta(m\pm 1)\,\mathrm{e}^{\mathrm{i}(m\pm 2)\phi}\frac{1}{\sqrt{\tilde{x}}}\left(\mp\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} + \frac{m\pm 1}{2\tilde{x}} - \frac{\gamma}{4(m\pm 1)}\beta^2\right) f_{nm}(\tilde{x}). \tag{29}$$

Because of equation (12) the effect of the radial operators in equations (29) acting on the reduced radial wavefunction is such that

$$o^-_{m+2} f_{nm}(\tilde{x}) \equiv \left(-\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} + t(\tilde{x}, m+1)\right) f_{nm}(\tilde{x}) \propto f_{n\,m+2}(\tilde{x}) \tag{30}$$

$$o^+_{m} f_{nm}(\tilde{x}) \equiv \left(\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} + t(\tilde{x}, m-1)\right) f_{nm}(\tilde{x}) \propto f_{n\,m-2}(\tilde{x}) \tag{31}$$

with

$$t(\tilde{x}, m) = \frac{m}{2\tilde{x}} - \frac{\gamma}{4m}\beta^2. \tag{32}$$

Alternatively, from the IHFM we can deduce expressions for the o^-_{m+2} and o^+_m operators as follows. In terms of $f_{nm}(\tilde{x})$, equation (25) turns out to be

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\tilde{x}^2} + \frac{1}{4}\frac{(m-1)(m+1)}{\tilde{x}^2} - \frac{\gamma}{4\tilde{x}}\right) f_{nm}(\tilde{x}) = -\frac{1}{4} f_{nm}(\tilde{x}) \tag{33}$$

which is an F-type factorizable equation in the sense of IH [1].

By means of the IHFM, equation (33) can be factorized into two equations completely equivalent to equations (30) and (31). Therefore, the operators o^-_{m+2} and o^+_m must satisfy

$$o^-_{m+2} f_{nm}(\tilde{x}) = \sqrt{1/4 + 2\eta_m}\, f_{n\,m+2}(\tilde{x}) \tag{34}$$

$$o^+_m f_{nm}(\tilde{x}) = \sqrt{1/4 + 2\eta_{m-2}}\, f_{n\,m-2}(\tilde{x}) \tag{35}$$

where

$$2\eta_m = \frac{\gamma^2}{16(m+1)^2}. \tag{36}$$

These operators connect the reduced radial eigenfunctions f_{nm} and $f_{n\,m+2}$, i.e. o^-_{m+2} transforms f_{nm} into $f_{n\,m+2}$ and o^+_{m+2} does the reverse. Thus, we conclude that the IH radial operators o^-_{m+2} and o^+_m are contained in the constants of motion $S_\pm$.

3. The $S_\pm$ operators and supersymmetric approach to the problem

We note from the radial Schrödinger equation (25) or (33), that to each fixed value of m there corresponds a Hamiltonian H_m, with the property

$$H_m = H_{-m}. \tag{37}$$

This means that

$$f_{nm} = f_{n|m|}. \tag{38}$$

As a consequence, in an mn-plane the levels with the same energy are symmetrically grouped around the n-axis. Consistently with this picture we observe that

$$o^-_{-m+2} = -o^+_m \tag{39}$$

which is valid for $m \neq 1$, as was noted in section 2. Then there is no operator o^-_1 (o^+_1) transforming $f_{n\,-1}$ into $f_{n\,1}$ ($f_{n\,1}$ into $f_{n\,-1}$). This is so because, in agreement with equation (37), the pair of Hamiltonians H_1 and H_{-1} are identical.

The grouping of the angular momentum levels allows us to relate the operators S^+ and S^- to the supercharges $Q^\pm_m$ in the supersymmetric approach to the problem.

By applying a_g to the state $\psi(\rho, \phi)$ the resulting radial operators are [12]

$$Q^+_m \equiv a_g(\rho, m) = \frac{\hbar}{\sqrt{2\mu}}\left(\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{m+\frac{1}{2}}{\rho} - \beta^2\rho\right) \tag{40}$$

$$Q^-_m \equiv a^\dagger_g(\rho, m+1) = \frac{\hbar}{\sqrt{2\mu}}\left(\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{m+\frac{1}{2}}{\rho} - \beta^2\rho\right) \tag{41}$$

which lead us to the pair of isospectral Hamiltonians of the problem,

$$Q^+_m Q^-_m = -\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + V^{(0)} - m\hbar\omega \tag{42}$$

$$Q^-_m Q^+_m = -\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + V^{(1)} - (m+1)\hbar\omega \tag{43}$$

where

$$V^{(0)} = V(\rho) + \frac{\hbar^2}{2\mu}\frac{m(m+1)+\frac{3}{4}}{\rho^2} \tag{44}$$

$$V^{(0)} = V(\rho) + \frac{\hbar^2}{2\mu}\frac{m^2 - \frac{1}{4}}{\rho^2} \tag{45}$$

Thus, we have shown that the operators $S_\pm$, given explicitly by equation (23), generate the radial ladder operators (30) and (31), as well as the pair of isospectral Hamiltonians (42) and (43) for the 2-DIHO.

4. Concluding remarks

We have shown that $S_\pm$ are ladder operators simply from their commutation relations with L_z. We have found explicitly how these operators are translated from the classical observables and their relation to the left and right circular annihilation operators. Moreover, we have found that the operators $S_\pm$ generate the IH radial factorization and the pair of isospectral Hamiltonians of the problem.

In [13] Lyman and Aravind have derived the Laplace–Runge–Lenz vector for the two-dimensional hydrogen atom from the corresponding supersymmetric radial operators. Also, Torres and Tepper [14] have recently derived the constants of motion S_x and S_y of the 2-DIHO from the radial super charges. However, these authors use the inverse procedure to that followed by us.

As a final remark, the 3-DIHO group of symmetry is well known [15]. Also, the radial Schrödinger equation has been factorized and raising and lowering operators have been derived [16]. The connection of supersymmetry to the constants of motion is the subject of a forthcoming report.

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